Basis for Derivation of Matrices for the Direct Stiffness Method

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Previous developments in the direct stiffness method are reviewed. The advantages of extending the definitions to make the method a variational approach are cited. The finite element formulation of the method of minimum potential energy is given. Explicit requirements of potential energy displacements are presented, and a criterion insuring monotonic convergence is developed. Illustrative displacements yielding matrices resulting in monotonic convergence are included. Available matrices are reviewed with respect to the extended developments.

Previous Developments

INITIAL efforts in the direct stiffness method were devoted to obtaining the capability to analyze low-aspect-ratio airplane wings. Levy[4] introduced the ideas of replacing the continuous structure by pieces, generating a stiffness matrix for each element, and summing stiffnesses. He proposed treating the wing as a collection of elementary beams and torque boxes. Turner et al.[17] refined this application by reducing the torque boxes to assemblies of triangular or rectangular slices. To improve the accuracy of the method for thin-wing analyses and to improve convergence characteristics, Melosh and Merritt[8] introduced new beam matrices.

Additional matrices have been developed for application of the method to other geometries. Melosh,[7] Dill and Ortega,[2] and Papenfuss[11] obtained matrices for rectangular plates. Turner[16] developed a matrix for a tapered spar. Grafton and Strome[3] proposed one for a symmetrically loaded conical shell element, and Martin[6] suggested one for a tetrahedron.

Other efforts have been directed toward generalizing the matrices for the analysis of large-deflection and buckling behavior. Matrices for the large-deflection analysis of heated structures have been presented by Turner et al.[18] Matrices for buckling analyses of beam and column networks have been given by Renton.[13]

Despite all these efforts, no one has defined a suitable basis for development of an "error-consistent" set of stiffness matrices. Consequently, application of the matrices developed has resulted in solutions that are sometimes too stiff, sometimes too soft, sometimes exhibit monotonic convergence as the number of nodes is increased, sometimes show oscillatory convergence, and sometimes apparently diverge.[10,11]

In the following paragraphs, these difficulties are removed by extending the requirements under which stiffness matrices are derived. In the first section, advantages of using the extremum variational theorems are reviewed, and the matrix statements and interpretation are given for stiffness solutions. The second section contains additional requirements for displacements in the potential energy approach, development of a convergence criterion, and a summary of displacement requirements. The third section provides stiffness matrices derived directly from displacement functions. The final sections include an evaluation of available matrices with respect to the extended requirements and a summary of developments of this paper.

1. Extremum Theorems

If approximate solutions are obtained using the extremum variational theorems of elasticity, bounds on mean displacement, stress, and strain can be determined. In principle, values of these elastic responses at a point can be generated. It can be proven that the strain energy of the solution must be greater than that of a minimum potential energy approximation and less than that of any minimum complementary energy approximation (see Synge,[15] pp. 98–117, or Prager,[12] pp. 3–5). Using this theorem and Green's function in elasticity, Prager has developed the means for bounding the elastic behavior. Benthem[1] has provided the interpretation of Prager's method for practical structures.

Errors associated with approximate solutions may be classified as idealization, discretization, and manipulation errors. Idealization errors are those involved in formulating a mathematical model of the structure. Examples of these errors are the errors associated with using flat surfaces to simulate curved ones, using pinned joints to simulate partially restrained ones, replacing geometric orthotropy with material orthotropy, or replacing varying depth elements with constant depth pieces. Discretization errors are those associated with replacing the continuous structure by one composed of finite elements. If fictitious cuts are made infinitely close together, the discretization error vanishes, assuming convergence. Manipulation errors are the round-off, truncation, and arithmetic errors incurred in performing the required calculations. The bounding theorems are applicable to defining limits of the discretization error.

In defining bounds, both the minimum potential and minimum complementary energy approximations are needed. Both analyses can follow the same sequence of operations. The two solutions are companion analyses, one the dual of the other. Because of this duality, most of the remainder of this paper will be concerned with only one approach: the potential energy formulation.

The Appendix presents the matrix formulation of the potential energy approach using a finite element representation. One point indicated in the Appendix merits special emphasis. Within the boundaries of an element, stresses are not defined uniquely by the potential energy solution. Some authors have found "stresses" by differentiating the displacements determined in the approximate solution. These must be regarded only as typical values. Because the assumed displacements are used only as a device to define energy (the stiffness coefficients), their influence on the answers is only indirect. The theory admits of an interpretation of stress in terms of some weighted mean over the element.

2. Displacement Requirements

Although all the necessary requirements for acceptable displacement functions are implicit in the theory presented in

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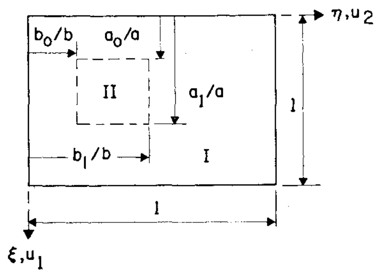

Fig. 1 Undivided panel

the Appendix, it is desirable to restate these requirements. Moreover, it is attractive to add some optional requirements: requirements that it is advantageous but not necessary to satisfy. These latter ones will be considered first.

Displacements conforming with the potential energy theorem will lead to solutions with definable discretization error. However, unless the functions are selected carefully, increasing the number of fictitious cuts will not necessarily provide an answer with smaller error. In the following development, a condition is defined on the basic matrix $[G]$ of displacement functions. This condition requires that the displacements of a subelement may be identical to those generated in it by displacements on the element before subdivision. This insures that the state of minimum potential energy of the original structure is a possible deformation state of the subdivided structure. Hence monotonic convergence is assured.

To particularize the discussion, consider the rectangular slice with nondimensional geometry shown in Fig. 1. This region will be called panel I. The smaller dotted region will be referred to as panel II. η and ξ are nondimensional coordinates and a and b true lengths of the sides along x and y, respectively.

On the basis of an assumed set of displacements, the displacements over panel I are defined. Since the equations are of the form (A1), an expression for displacements over panel II is

$$\{u\} = [G]\{v_{\mathrm{I}}\} \tag{1}$$

The v_{I} are the deformations of the nodes bounding panel I.

The displacements developed over panel II also can be written directly in the coordinate system of Fig. 2:

$$\{\bar{u}\} = [\bar{G}]\{v_{\mathrm{II}}\} \tag{2}$$

The elements of $\bar{G}$ are the same as those of G but involve barred coordinates. The form of the elements of both of these matrices can be seen by nondimensionalizing Eq. (6).

Expressions (1) and (2) can be made more consonant by using Eq. (A1) to define v_{II} in terms of v_{I} and by transforming coordinates. The relevant relations are

$$\{v_{\mathrm{II}}\} = [H(\xi_{\mathrm{II}},\eta_{\mathrm{II}})]\{v_{\mathrm{I}}\} = [H]\{v_{\mathrm{I}}\} \tag{3}$$

$$\bar{\xi} = (a\xi - a_0)/(a_1 - a_0) \qquad \bar{\eta} = (b\eta - b_0)/(b_1 - b_0) \tag{4}$$

The a_i and b_i describe the nondimensional distances shown in Fig. 1. The H matrix is generated by substituting the coordinates of the nodes of panel II in rows of the G matrix. The unbarred coordinates are used.

Equations (1) and (2) will be identical if and only if

$$[g][H] = [G]$$

$$[g] = [G(\bar{\xi}[\xi],\bar{\eta}[\eta])] = \left[G\left(\left\{\frac{a\xi - a_0}{a_1 - a_0}\right\},\left\{\frac{b\eta - b_0}{b_1 - b_0}\right\}\right)\right] \tag{5}$$

This equation requires that the displacements over panel II be the same when the panel is undivided as in Fig. 1 or subdivided as in Fig. 2. To permit arbitrary subdivision, Eq. (5) must be satisfied for any location and side ratio of panel II within panel I.

In general, the transformation, Eq. (4), will involve translations, rotations, and scaling of coordinates. The form of Eq. (5) is independent of the specific nature of the transformation and will not change if other than a rectangular slice were used for its development.

For simple functions, it is easy to visualize satisfaction of (5) without calculation. For example, if the displacements can be visualized as a plane surface over the panel I (as it can be for the triangular slice matrix), it is clear that Eq. (5) will be satisfied for translations, rotations, and scaling.

The proof that monotonic convergence must result is simple. If (5) is satisfied, the total energy of a subdivided region can be identical to that of the undivided region. Since the minimum energy solution is selected in each analysis, the subdivided region analysis cannot result in a higher total energy solution than that of the undivided panel, i.e., the process must exhibit monotonic convergence.

The criterion is dependent only upon the assumed displacements. It is independent of the nature of the material and geometric orthotropy and the form of the generalized displacements. It can be applied to any number of spatial coordinates, to any geometry, and to the structure with body forces. Essentially, it insures monotonic convergence as long as the structural idealization is not redefined as the structure is subdivided.

The criterion, however, does not insure convergence to the exact solution. Since it generally is desired to obtain only an approximation to the answer, convergence to the exact solution is of secondary interest and will be of no further concern here. The interested reader may refer to Synge,[15] pp. 209–213, for an example of a proof of convergence to the exact answer. This proof applies to the triangular slice stiffness matrix.

In addition to choosing displacements to conform with Eq. (5), it is advantageous to include the complete set of rigid-body displacements. As a consequence of including these displacements, columns and rows of the unconstrained stiffness matrix will satisfy the macroscopic equations of equilibrium for the element. This characteristic provides a basis for performing checks on manipulation errors.

It can be proved that the macroscopic equilibrium equations must be satisfied by considering the strain energy. If rigid states are included in forming the stiffness matrix, the strain energy must vanish for a set of rigid displacements, since rigid states involve no strains. For this same reason,

$$[v_r][K]\{v_e\} = 0$$

where v_r are rigid displacement components and v_e any elastic deformations. Choosing the vector v_e in the form $(0, 0, \ldots 1, \ldots 0)^T$ and comparing components of v_r with the force coefficients in the macroscopic equilibrium equations, the theorem can be proved. In rectangular Cartesian coordinates, for example, translations correspond with force equilibrium and rotations with moment equilibrium equations.

Since rigid states involve no potential energy, a heuristic argument suggests that selecting rigid states in deference to elastic will result in a lower energy and hence a better approximate solution. It also is noted that, although rigid states are desirable, their inclusion is not sufficient to insure convergence to the exact solution.

In conclusion, in accordance with the Appendix, the assumed displacements must comply with the following requirements:

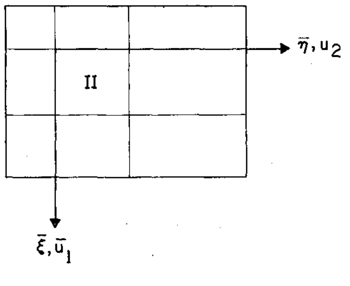

Fig. 2 Subdivided panel

1) They must be continuous over the elements. (They need not have continuous derivatives.)

2) They must maintain continuity with displacements of adjacent elements. When nodal displacements are selected as generalized displacements, this requirement is satisfied easily. The displacements along any side of the element are selected so that they depend only on the displacements at the nodes bounding the side.

3) The displacement function must be a linear function of the generalized displacements. This is necessary so that the load-displacement equations will be linear. In the displacement expression, as a consequence, the coefficient of the nodal displacement must be nondimensional, and the coefficient of nodal rotation must involve a length unit to satisfy dimensional requirements.

Optional requirements for displacement functions discussed in this section include the following:

4) They should include rigid body states as independent behavior states. Rigid states can be defined by solutions of the homogeneous (inextensional) strain equations. A check on whether or not these states are included (necessary) can be made by showing that the forces in each column of the stiffness matrix satisfy the macroscopic equations of equilibrium for the element.

5) They should conform with the equation of the form (5).

All of these stipulations are independent of element geometry, material characteristics, and the smallness of strains and displacements. A suitable displacement function can be used with any definition of the strain energy.

3. Examples

As a first example of selection of displacements for application of the extended stiffness method, consider the rectangular slice shown in Fig. 3. The origin of the coordinates is chosen at the center of gravity of the slice for convenience. The sides lie parallel to the x and y directions and are of length $2a$ and $2b$, respectively. Nodes are numbered clockwise, starting in the upper left-hand corner. The Lagrange interpolation formula in two dimensions[14] leads directly to the displacement expressions:

$$\begin{aligned} 4abu &= (x-a)(y-b)u_1 - (x-a)(y+b)u_2 + \\ &\qquad (x+a)(y+b)u_3 - (x+a)(y-b)u_4 \\ 4abv &= (x-a)(y-b)v_1 - (x-a)(y+b)v_2 + \\ &\qquad (x+a)(y+b)v_3 - (x+a)(y-b)v_4 \end{aligned} \tag{6}$$

where u and v are displacements in the x and y directions, respectively. Displacement components with subscripts apply to nodal displacements (the generalized displacements in this example).

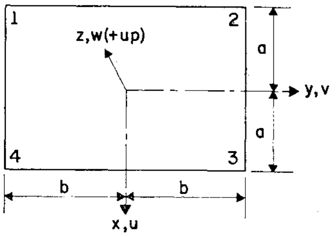

Fig. 3 Rectangular geometry

It is easy to show that (6) satisfies all the requirements of Sec. 2. The rigid-body displacements are translation in the x direction, translation in y, and rotation about the z axis.

If the stress-strain equations are given by the equations

$$\begin{Bmatrix} \sigma_{xx} \\ \sigma_{yy} \\ \sigma_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{21} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{44} \end{bmatrix} \begin{Bmatrix} \epsilon_{xx} \\ \epsilon_{yy} \\ \epsilon_{xy} \end{Bmatrix} \tag{7}$$

then one is led to the stiffness matrix given in Table 1. It is observed that the displacement function is a hyperbolic paraboloid. Edges of the slice coincide with the line elements of the paraboloid so that displacements along any side define a straight line.

As a second example, consider Fig. 3 as depicting a rectangular plate in planform. Let displacements be taken as

$$\begin{aligned} 32w = {} & X_mY_m\{2(X_mY_m - X_pY_p)w_1 - \\ & 4X_pX_m(w_{,1x} + w_1) - 4Y_pY_m(w_{,1y} + w_1)\} + \\ & X_mY_p\{2(X_mY_p - X_pY_m)w_2 + \\ & 4X_pX_m(w_{,2x} + w_2) - 4Y_pY_m(w_{,2y} - w_2)\} + \\ & X_pY_p\{2(X_pY_p - Y_mX_m)w_3 + \\ & 4X_pX_m(w_{,3x} - w_3) + 4Y_pY_m(w_{,3y} - w_3)\} + \\ & X_pY_m\{2(X_pY_m - X_mY_p)w_4 - \\ & 4X_pY_m(w_{,4x} - w_4) + 4Y_pY_m(w_{,4y} + w_4)\} \\ & X_p = (x+a)/a \qquad X_m = (x-a)/a \\ & Y_p = (y+b)/b \qquad Y_m = (y-b)/b \end{aligned} \tag{8}$$

where w is the displacement normal to the plate and the comma denotes partial differentiation.

It can be shown that (8) satisfies all the necessary and optional requirements for displacement functions. Rigid displacements include translation normal to the plate and rotations about the x and y axes. Equation (8) results in zero curvature for any linear combination of these states. It is interesting to note that Eq. (8) satisfies the homogeneous equilibrium equation of plates identically.

Table 1 Rectangular slice stiffness matrix

$$[K] = \begin{array}{cc} u_i & v_i \\ \left[\begin{array}{c} K_{11} \\ K_{21} \end{array}\right. & \left.\begin{array}{c} K_{21}{}^T \\ K_{22} \end{array}\right] \end{array}$$

$a_{11} = tA_{11}b/6a$
$a_{22} = tA_{22}a/6b$
$a_{12} = tA_{21}/4$

t = slice thickness
$\bar{a}_{44} = tA_{44}a/6b$
$\dot{a}_{44} = tA_{44}b/6a$
$a_{44} = tA_{44}/4$

$$[K_{11}] = \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \left[\begin{array}{c} 2a_{11} + 2\bar{a}_{44} \\ a_{11} - 2\bar{a}_{44} \\ -a_{11} - \bar{a}_{44} \\ -2a_{11} + \bar{a}_{44} \end{array}\right. & \begin{array}{c} \\ 2a_{11} + 2\bar{a}_{44} \\ -2a_{11} + \bar{a}_{44} \\ -a_{11} - \bar{a}_{44} \end{array} & \begin{array}{c} \\ \\ 2a_{11} + 2\bar{a}_{44} \\ a_{11} - 2\bar{a}_{44} \end{array} & \left.\begin{array}{c} \text{symmetric} \\ \\ \\ 2a_{11} + 2\bar{a}_{44} \end{array}\right] \end{array}$$

$$[K_{22}] = \begin{array}{cccc} v_1 & v_2 & v_3 & v_4 \\ \left[\begin{array}{c} 2a_{22} + 2\dot{a}_{44} \\ -2a_{22} + \dot{a}_{44} \\ -a_{22} - \dot{a}_{44} \\ a_{22} - 2\dot{a}_{44} \end{array}\right. & \begin{array}{c} \\ 2a_{22} + 2\dot{a}_{44} \\ a_{22} - 2\dot{a}_{44} \\ -a_{22} - \dot{a}_{44} \end{array} & \begin{array}{c} \\ \\ 2a_{22} + 2\dot{a}_{44} \\ -2a_{22} + \dot{a}_{44} \end{array} & \left.\begin{array}{c} \text{symmetric} \\ \\ \\ 2a_{22} + 2\dot{a}_{44} \end{array}\right] \end{array}$$

$$[K_{21}] = \begin{array}{cccc} u_1 & u_2 & u_3 & u_4 \\ \left[\begin{array}{c} a_{12} + a_{44} \\ -a_{12} + a_{44} \\ -a_{12} - a_{44} \\ a_{12} - a_{44} \end{array}\right. & \begin{array}{c} a_{12} - a_{44} \\ -a_{12} - a_{44} \\ -a_{12} + a_{44} \\ a_{12} + a_{44} \end{array} & \begin{array}{c} -a_{12} - a_{44} \\ a_{12} - a_{44} \\ a_{12} + a_{44} \\ -a_{12} + a_{44} \end{array} & \left.\begin{array}{c} -a_{12} + a_{44} \\ a_{12} + a_{44} \\ a_{12} - a_{44} \\ -a_{12} - a_{44} \end{array}\right] \end{array}$$

Table 2 Rectangular plate stiffness matrix

$[K] = [K_1] + [K_2] \qquad a_{11} = (2b/3a^3)A_{11} \qquad a_{22} = (2a/3b^3)A_{22}$

K_1 matrix

$$\left[\begin{array}{cccccccccccc}
w_1 & w_{,1x} & w_{,1y} & w_2 & w_{,2x} & w_{,2y} & w_3 & w_{,3x} & w_{,3y} & w_4 & w_{,4x} & w_{,4y} \\
6a_{11} + 6a_{22} & & & & & & & & & & & \\
6a_{11} & 8a_{11} & & & & & & \text{symmetric} & & & & \\
6a_{22} & 0 & 8a_{22} & & & & & & & & & \\
3a_{11} - 6a_{22} & 3a_{11} & -6a_{22} & 6a_{11} + 6a_{22} & & & & & & & & \\
3a_{11} & 4a_{11} & 0 & 6a_{11} & 8a_{11} & & & & & & & \\
6a_{22} & 0 & 4a_{22} & -6a_{22} & 0 & 8a_{22} & & & & & & \\
-3a_{11} - 3a_{22} & -3a_{11} & -3a_{22} & -6a_{11} + 3a_{22} & -6a_{11} & -3a_{22} & 6a_{11} + 6a_{22} & & & & & \\
3a_{11} & 2a_{11} & 0 & 6a_{11} & 4a_{11} & 0 & -6a_{11} & 8a_{11} & & & & \\
3a_{22} & 0 & 2a_{22} & -3a_{22} & 0 & 4a_{22} & -6a_{22} & 0 & 8a_{22} & & & \\
-6a_{11} + 3a_{22} & -6a_{11} & 3a_{22} & -3a_{11} - 3a_{22} & -3a_{11} & 3a_{22} & 3a_{11} - 6a_{22} & -3a_{11} & 6a_{22} & 6a_{11} + 6a_{22} & & \\
6a_{11} & 4a_{11} & 0 & 3a_{11} & 2a_{11} & 0 & -3a_{11} & 4a_{11} & 0 & -6a_{11} & 8a_{11} & \\
3a_{22} & 0 & 4a_{22} & -3a_{22} & 0 & 2a_{22} & -6a_{22} & 0 & 4a_{22} & 6a_{22} & 0 & 8a_{22}
\end{array}\right]$$

$a_{12} = (2/ab)A_{21} \qquad a_{33} = (2/15ab)A_{33}$

K_2 matrix

$$\left[\begin{array}{cccccccccccc}
w_1 & w_{,1x} & w_{,1y} & w_2 & w_{,2x} & w_{,2y} & w_3 & w_{,3x} & w_{,3y} & w_4 & w_{,4x} & w_{,4y} \\
a_{12} + 2a_{33} & & & & & & & & & & & \\
a_{12} + 3a_{33} & 8a_{33} & & & & & & \text{symmetric} & & & & \\
a_{12} + 3a_{33} & 6a_{12} & 8a_{33} & & & & & & & & & \\
-a_{12} - 2/a_{33} & -a_{12} - 3a_{33} & -3a_{33} & a_{12} + 2/a_{33} & & & & & & & & \\
-a_{12} - 2a_{33} & -8a_{33} & 0 & -a_{12} + 3a_{33} & 8a_{33} & & & & & & & \\
3a_{33} & 0 & -2a_{33} & -a_{12} - 3a_{33} & -6a_{12} & 8a_{33} & & & & & & \\
a_{12} + 2/a_{33} & 3a_{33} & 3a_{33} & -a_{12} - 2/a_{33} & -3a_{33} & a_{12} + 3a_{33} & a_{12} + 2/a_{33} & & & & & \\
-3a_{33} & 2a_{33} & 0 & 3a_{33} & -2a_{33} & 0 & -a_{12} - 3a_{33} & 8a_{33} & & & & \\
-3a_{33} & 0 & 2a_{33} & a_{12} + 3a_{33} & 0 & -8a_{33} & -a_{12} - 3a_{33} & 6a_{12} & 8a_{33} & & & \\
-a_{12} - 2/a_{33} & -3a_{33} & -a_{12} - 3a_{33} & a_{12} + 2/a_{33} & 3a_{33} & -3a_{33} & -a_{12} - 2/a_{33} & a_{12} + 3a_{33} & 3a_{33} & a_{12} + 2/a_{33} & & \\
3a_{33} & -2a_{33} & 0 & -3a_{33} & 2a_{33} & 0 & a_{12} + 3a_{33} & -8a_{33} & 0 & -a_{12} - 3a_{33} & 8a_{33} & \\
-a_{12} - 3a_{33} & 0 & -8a_{33} & 3a_{33} & 0 & 2a_{33} & -3a_{33} & 0 & -2a_{33} & a_{12} + 3a_{33} & -6a_{12} & 8a_{33}
\end{array}\right]$$

Table 3 Available stiffness matrices

Element	Source	Geometry	Idealization	Displacements along side	Continuity over piece	Monotonic convergence
Torque box, $T1$	4	Quadrilateral	Uniform GJ tube	Linear	Yes	Yes
Rod, $R1$	17	Line	Short column	Linear	Yes	Yes
Beam, $B1$	17	Rectangle	Uniform EI, GA beam	Cubic	Yes	Yes
Beam, $B2$	8	Rectangle	Uniform EI, GA composite	Quadratic	Yes	Yes
Beam, $B3$	8	Rectangle	Uniform GA shear web	Linear	Yes	Yes
Beam, $B4$	16	Trapezoid	Tapered EI, GA composite	Linear	Yes	Yes
Slice, $S1$	17	Triangle	Uniform gage	Linear	Yes	Yes
Slice, $S2$	17	Rectangle	Uniform gage	Quadratic	Yes	No
Slice, $S3$	Here	Rectangle	Uniform gage	Linear	Yes	Yes
Plate, $P1$	7	Rectangle	Flat thin plate	Cubic	No	No
Plate, $P2$	2	Rectangle	Flat thin plate	Cubic	Yes	No
Plate, $P3$	11	Rectangle	Flat thin plate	Cubic	No	No
Plate, $P4$	Here	Rectangle	Flat thin plate	Cubic	Yes	Yes
Solid, $D1$	6	Tetrahedron	Three-dimensional solid	Planar	Yes	Yes
Shell, $L1$	3	Conical piece	Closed shell segment	Cubic	Yes	Yes

If the moment-curvature relationship is given by

$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{21} & 0 \\ A_{21} & A_{22} & 0 \\ 0 & 0 & A_{33} \end{bmatrix} \begin{Bmatrix} K_1 \\ K_2 \\ \tau \end{Bmatrix} \tag{9}$$

then the stiffness matrix obtained from (9) is that given in Table 2. The displacement function, Eq. (9), was obtained by modifying an osculating interpolation function.

In both of these derivations, developments have started with a displacement function. This approach greatly simplifies the derivation.

4. Evaluation of Existing Stiffness Matrices

A summary of the features of interest of matrices available in the literature is given in Table 3. This table defines the elements, their geometry and idealized structure, and the nature of assumed displacements along the sides. All the matrices but $P1$, $P3$, $S3$, and $L1$ (see Table 3) were derived by finding displacements that would satisfy both the microscopic equilibrium and the compatibility equations. Most matrices were derived assuming isotropic materials. This objection is removed easily by an exercise in algebra and elementary geometry (see Lunder[5]). All matrices cited are based on polynomial displacement functions. This will be seen to be fortuitous in complying with the second requirement of Sec. 2 in composite structure analyses.

The last two columns of the table indicate the compliance of displacements with requirements 1, 2, and 5 of Sec. 2. All matrices are based on displacements conforming with requirements 3 and 4. The evaluation of the matrices with respect to their ability to maintain displacement continuity across the boundaries implies that the neighboring element is of the same form.

All the previously developed rectangular plate matrices are unacceptable. The old rectangular slice matrix fails to maintain displacement continuity across the edges. The new matrices included here should supplant the old ones.

The second column of Table 4 indicates some structural systems that can be analyzed using only the acceptable matrix given in the first column. Recognizing that most of these matrices imply some type of inextensional behavior, the analyst will impose appropriate constraints when his analysis is of folded structures.

The last column of Table 4 includes comments on the suitability of the acceptable matrices for analysis of a composite system. The type of assumed displacements forming the basis for this class of analysis is that which is linear on a side. Matrices expressed by higher-order polynomials must be constrained to preserve continuity with adjacent ones. The formulation then coincides with Eq. (A11).

These elements retain displacement continuity when they are adjacent to each other. Other positioning may be inadmissible. For example, if a rod is connected across the diagonal of the rectangular slice $S3$, continuity is not maintained.

For the line-element matrices, $R1$, $B1$, $B2$, $B3$, and $B4$, continuity must be considered for both the ends and sides of the line. This is so since the continuity must be viewed from a microscopic aspect. When used with other line elements, however, continuity only over the ends is required. These facts provide a basis for the previous convergence discussion by Melosh and Merritt.[8]

Other classes of analysis also can be defined which permit a rigorous minimum potential energy solution to be executed with stiffness matrices. Quadratic and cubic displacement polynomials, for example, are useful choices because of the assumed displacements in available matrices. In these classes of analysis, the matrices involving linear displacements along a side would be unsuitable.

5. Minimum Complementary Energy Analyses

In this paper, emphasis has been placed on the minimum potential energy approach. To many of the given statements, however, there corresponds a dual statement of the complementary energy approach. To every relation and statement in the Appendix, a corresponding dual exists. To each of the requirements in Sec. 2, except requirement 4, a dual exists.

Since flexibility matrices are summed in the dual, none of the stiffness matrices are interpretable directly as members of

Table 4 Suitability of acceptable matrices

Element	Description of "self-system"	Class I composite structure analysis suitability
$T1$	Variable stiffness crankshaft	Directly suitable
$R1$	Truss	Directly suitable
$B1$	Framework	Suitable when displacements are constrained
$B2$	Framework	Unsuitable
$B3$	Framework	Directly suitable
$B4$	Framework	Directly suitable
$S1$	Stepped thickness sheet	Directly suitable
$S3$	Stepped thickness sheet	Directly suitable
$P4$	Stepped thickness plate	Suitable when displacements are constrained
$D1$	Conical and degenerate conical shells	Suitable when displacements are constrained
$L1$	Three-dimensional solids	Directly suitable

the dual. The line element matrices, however, imply stresses that satisfy the requirements of the complementary approach. Since these can be regarded as satisfying both sets of requirements, they can be considered exact representations.

In defining bounds, both energy approaches must be used. An apparent incompatibility exists in satisfying exactly the boundary conditions for both methods in a given problem. This is resolved by reinterpreting the idealization error. Over that part of the structure where displacements are defined, displacement boundary conditions are idealized so that they can be matched exactly with assumed displacements. Similarly, over that part of the structure where stresses are prescribed, stress boundary conditions are idealized so that they can be matched exactly with assumed stress functions.

6. Summary of Significant Developments

The salient features of the preceding sections are summarized as follows:

1) The stiffness method definition has been extended to include requirements on displacements so that the method is a rigorous potential energy formulation of the elasticity problem. The extended definition, in combination with its dual, provides bounds on the strain energy. These analyses, moreover, can lead to the determination of bounds on strain, stress, and displacement over an element of the structure.

2) A criterion [Eq. (5)] has been developed to determine whether or not an assumed displacement function will provide a stiffness matrix that will lead to monotonic convergence of results with network subdivision. Although only a sufficient condition, the ease with which functions can be generated conforming with this requirement suggests that it will be adequate.

3) Nine of 13 previously developed stiffness matrices have been found to be suitable for a minimum potential energy analysis. Eight are suitable for a composite structure analysis (Table 4). Suitable new matrices have been generated for a rectangular slice and plate (Tables 1 and 2).

Appendix: Finite Element Form of the Method of Minimum Potential Energy

Assume a set of displacements as a linear function of some arbitrary parameters: the generalized coordinates. Displacements must be continuous over the element, preserve displacement continuity across element boundaries, and match displacement boundary conditions, but they need not satisfy the Cauchy equilibrium equations. Then, the displacements can be written in the form

$$\{u\} = [D]\{v\} \tag{A1}$$

where u_i are the displacements and v_j, $j = 1, 2, \ldots m$ are the generalized displacements.

Using (1) and the strain-energy relation, the strain energy can be written in terms of the generalized deformations. Thus, taking σ_i as stress,

$$\{\sigma\} = [A][B]\{u\} \tag{A2}$$

where the A matrix consists of material constants and B involves differential operators; the strain energy is given by

$$U = \tfrac{1}{2}\int_{V_0} [u][B^T][A][B]\{u\}\; dV_0 \tag{A3}$$

where V_0 designates volume. Substituting (A1) in (A3), the strain energy can be written in a more useful form:

$$U = \tfrac{1}{2}\int_{V_0}[v][D^T][B^T][A][B][D]\{v\}dV_0 \tag{A4}$$

$$u = \tfrac{1}{2}[v][K]\{v\} \tag{A5}$$

$$[K] = \int_{V_0} [D^T][B^T][A][B][D]dV_0$$

The matrix K is called the stiffness matrix. Since A is positive semidefinite matrix for any real material, so is the stiffness matrix.

Now, the potential energy functional is written in terms of the displacements and prescribed quantities. The functional is defined by

$$V = U - \int_{S_t}[T]\{u\}dS_t - \int_{V_0}[X]\{u\}dV_0 \tag{A6}$$

where the T_i are surface tractions, X_i body forces, and S_t denotes that part of the surface over which tractions are prescribed. The T_i are defined in terms of stresses on the surface by the operator N, whose terms are components of the surface normal, i.e.,

$$\{T\} = [N]\{\sigma_t\} \tag{A7}$$

The subscript designates prescribed quantities, i.e., stresses in this case. Using (A7) and (A1), (A6) becomes

$$V = U - \int_{S_t}[v][D^T][N]\{\sigma_t\} - \int_{V_0}[v][D^T]\{X\}dV_0 \tag{A8}$$

The generalized force-deformation equations are found by minimizing the potential, permitting only the deformations to vary. From (A5) and (A8), this operation gives

$$\{P\} = [K]\{v\}$$

$$\{P\} = \left\{\frac{\partial}{\partial v}\left[\int_{S_t} [v][D^T][N]\{\sigma_t\}dS_t + \int_{V_0}[v][D^T]\{X\}dV_0\right]\right\} \tag{A9}$$

Since K relates generalized forces and deformations, it is justified in being called a stiffness matrix. The generalized forces are seen to consist of weighted integrals of the prescribed stresses and body forces. Thus, the theory defines the forces corresponding to a prescribed set of stresses; no new approximation is necessary.

Displacement boundary conditions are satisfied by specifying some of the v_j. Assuming that those specified are the first n v_j $(n < m)$, then (A9) takes the form, omitting body forces,

$$\{\bar{P}\} = [K_{22}]\{v_{m-n}\}$$

$$\{\bar{P}\} = \left\{\frac{\partial}{\partial v_{m-n}} \int_{S_t} [v_{m-n}][D_{m-n}{}^T][N_{m-n}]\{\sigma_t\}dS_t\right\} - [K_{21}]\{v_n\} \tag{A10}$$

$$[K] = \begin{bmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{bmatrix}$$

$$[N] = [N_n\, N_{m-n}] \qquad [D] = [Dv\, D_{m-n}]$$

where the subscripts indicate the number of components.

Equations similar to (A10) also are obtained if the v_{n-m} are any linear function of v_n. Thus, if

$$\begin{aligned} \{v\} &= [L]\{v_n\} \\ \{\bar{P}\} &= [L^T][K][L]\{V_n\} \qquad \text{(A11)} \\ \{\bar{P}\} &= \left\{\frac{\partial}{\partial v_n} \int_{S_t} [v_n][L^T][D^T][N]\{\sigma_t\}dS_t\right\} \end{aligned}$$

Having found v_j from (A9), (A10), or (A11), the assumed displacements are found from (A1). Alternate assumptions of the form (A1) can lead to the same generalized force-displacement equations for a given problem, however. Therefore, use of (A1) and (A2) in an approximate solution determines only one of an infinity of possible stress states that are meaningful. Moreover, stresses so determined, in general, will not satisfy the microscopic equations of equilibrium.

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Interception of High-Speed Target by Beam Rider Missile

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The trajectories of a line-of-sight beam rider missile are determined for the case where the target is moving at suborbital speeds and the missile speed is smaller by a factor of 2 or 3. Of primary interest are the conditions under which interception may take place: the minimum target engagement range and interception range as functions of missile-to-target speed ratio and missile normal acceleration capability. The results of this analysis show that interception will take place for an arbitrary speed ratio with a reasonable value of lateral acceleration if the range at which the missile engages the target can be made sufficiently large.

Nomenclature

r,θ = plane polar coordinates in reference frame of Fig. 1
x,y = rectangular coordinates in reference frame of Fig. 1
t = missile flight time
V = speed
R = crossover range defined in Fig. 1
c = defined in Eq. (4)
u = missile lead angle
γ = missile flight path angle
τ = missile-to-target speed ratio
a_n = lateral missile acceleration

Superscripts

$\cdot$ = differentiation with respect to time
$'$ = differentiation with respect to θ
$*$ = intercept values at maximum value of y_m

Subscripts

m = missile values
t = target values
L = values at missile launch
I = values at interception

I. Introduction

A MISSILE guided by the so-called line-of-sight beam rider method has been shown to be capable of intercepting conventional-type targets, such as manned aircraft at moderate range, if the missile has a speed advantage.

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